

Algebraic Thinking (A)

- A01. Basics: Elementary number sentence basics errors (including family of facts.) , solving for a single variable using family of facts relationships, and alternate formats for multiplying, dividing, and grouping numbers
- A02. Error connecting a short verbal description of a relationship to the relationship's number sentence
- A03. Ratio and proportion problems
- A04. Pre-algebra level number sentence errors solving single-variable equations involving including multi-step problems requiring combining like terms, expanding across groupings, adding, subtracting, multiplying, or dividing by a constant.

A01. Basics

Algebra. Calculus. Both of them start in first grade. Surprised?

Algebra depends on these number sentence basics learned in grades 1 to 5. Calculus depends on algebra. Doing well in advanced math classes depends on learning these basics.

Ordering and grouping numbers. Answer “yes” or “no”

If	then does	Yes or No
$3 + 4 = 7$	$4 + 3 = 7 ?$	
$4 - 3 = 1$	$3 - 4 = 1 ?$	
$3 \times 4 = 12$	$4 \times 3 = 12 ?$	
$8 \div 2 = 4$	$2 \div 8 = 4 ?$	

Two answers are “no.” For adding and multiplying, any order works:

$$4 + 3 + 6 = 4 + 6 + 3 = 3 + 4 + 6 = 3 + 6 + 4 = 6 + 4 + 3 = 6 + 3 + 4$$

$$4 \times 3 \times 6 = 4 \times 6 \times 3 = 3 \times 4 \times 6 = 3 \times 6 \times 4 = 6 \times 4 \times 3 = 6 \times 3 \times 4$$

But order matters for subtraction and division.

What happens when the signs are mixed?

If you see	think: \times and \div first	so the answer is	Wrong! Wrong!
$4 + 3 \times 6 =$	$4 + (3 \times 6) =$	22	Not 42
$16 \div 2 + 6 =$	$(16 \div 2) + 6 =$	14	Not 2
$7 \times 3 - 2 =$	$(7 \times 3) - 2 =$	19	Not 7

Multiplication and division have “dibs” on being first.

“Hold on,” says addition and subtraction, “how can I be first?”

If you see	Think “parenthesis first”	so the answer is	Wrong! Wrong!
$(4 + 3) \times 6 =$	$7 \times 6 =$	42	Not 22
$16 \div (2 + 6) =$	$16 \div 8 =$	2	Not 12
$7 \times (3 - 2) =$	$7 \times 1 =$	7	Not 19

Suppose you saw: $4 \times (9 - 4) + 8 \div 4 - 3 \times 6 =$

Think: Parenthesis first (sometimes “parenthesis” is called “brackets.”)

Then: Multiplication and division second.

And poor: Adding and subtracting happen last.

So: Parenthesis first $4 \times 5 + 8 \div 4 - 3 \times 6 =$

Now \times and \div $20 + 2 - 18 =$

Then $+$ and $-$ $22 - 18 = 4$

Different ways to show multiplying and dividing

The very same operation can be show in different ways.

For example:	could be shown as	or	
$4 \times (6 - 2) =$	$4(6 - 2) =$	$4 \times 6 - 4 \times 2 =$	Remember, multiply first.
$a \times (b + c) =$	$a(b + c) =$	$ab + ac =$	Same as line above but now letters stand for numbers.
$7 \times 2 - 5 \times 2 =$	$2(7 - 5) =$	$2 \times (7 - 5) =$	The “2” is part of both multiplications.
$8 \div 4 =$	$\frac{8}{4} =$	$4 \sqrt{8} =$	Answer to all is 2.
$x \div y =$	$\frac{x}{y} =$	$y \sqrt{x} =$	Same as line above but now letters stand for numbers.

Perhaps, in your school, your teacher may always show questions the same way. But the time will come to “Show ‘em you know!” Your teacher will not write that test. Learn to recognize division and subtraction with letters or numbers these different ways.

Fact Families: add and subtract and multiply and divide

How many ways can the numbers 0 to 10 be added to themselves? As a hint, here is the first row. Testman counted the columns below.

$0+0=0$	$0+1=$	$0+2=2$	$0+3=3$	$0+4=4$	$0+5=5$	$0+6=6$	$0+7=7$	$0+8=8$	$0+9=9$	$0+10=10$
1	2	3	4	5	6	7	8	9	10	11

The very last row will be:

$10+0=10$	$10+1=11$	$10+2=22$	$10+3=13$	$10+4=14$	$10+5=15$	$10+6=16$	$10+7=17$	$10+8=18$	$10+9=19$	$10+10=20$
1	2	3	4	5	6	7	8	9	10	11

Eleven rows of eleven is one hundred twenty-one (also written 121 but you knew that.) Testman is picking one family: $8 + 5 = 13$. Call $8 + 5 = 13$ "YOU." Now, here are this family's brother and sister:

You	Your brother	Your sister
$8 + 5 = 13$	$13 - 5 = 8$	$13 - 8 = 5$

OK. Easy, eh? When the time comes to "Show 'em you know!" you need to be able to fill in all the missing places in the table below.

You	Your brother	Your sister
	$13 - 5 = 8$	
		$17 - 8 = 9$
$6 + 0 = 6$		
		$11 - a = 4$
	$m - n = 0$	
$x + 7 = 14$		

Here is the table when finished:

You	Your brother	Your sister
$5 + 8 = 13$	$13 - 5 = 8$	$13 - 8 = 5$
$9 + 8 = 17$	$17 - 9 = 9$	$17 - 8 = 9$
$6 + 0 = 6$	$6 - 6 = 0$	$6 - 0 = 6$
$a + 4 = 11$	$11 - 4 = a$	$11 - a = 4$
$n + 0 = m$	$m - n = 0$	$m - 0 = n$
$x + 7 = 14$	$14 - x = 7$	$14 - 7 = x$

If that seems hard, then you need practice. Each of those 121 different families must be known and known well. By the end of second grade, filling in this table needs to be done quickly and right. Yes, end of second grade.

Multiplication has the same number of families. Call $8 \times 5 = 40$ "YOU." This time, have an uncle and an aunt:

You	Your uncle	Your aunt
$8 \times 5 = 40$	$40 \div 5 = 8$	$40 \div 8 = 5$

By the end of third grade, when the time comes to "Show 'em you know!" you need to be able to fill in all the missing places in the table below.

You	Your uncle	Your aunt
	$40 \div 5 = 8$	
		$72 \div 8 = 9$
$6 \times 3 = 18$		
		$11 \div a = 4$
$z \times 7 = 14$		

Here is the table when finished:

You	Your brother	Your sister
$5 \times 8 = 40$	$40 \div 5 = 8$	$40 \div 8 = 5$
$9 \times 8 = 72$	$72 \div 9 = 9$	$72 \div 8 = 9$
$6 \times 3 = 18$	$18 \div 3 = 6$	$18 \div 6 = 3$
$a \times 4 = 11$	$11 \div 4 = a$	$11 \div a = 4$
$z \times 7 = 14$	$14 \div z = 7$	$14 \div 7 = z$

You need practice on these families until filling in the table seems easy. Each of those 121 different families must be known and known well. By the end of third grade, “Show ‘em you know!” tests will expect you to know these families well.

Errors A01 and A02 are part of what is called “pre-algebra.” Pre-algebra skills like this are taught between kindergarten and grade 7. These very same family relationships will carry into the REAL algebra.

When letters stand for numbers

In the table above, letters like x, a, and z stand for numbers. Wait! If a “z” stands for a number, say 21, why not just use the 21. That is a very logical question.

The z does not stand for just one number. The z stands for all numbers. So suppose you see an equation $z + 9 = y$. Here are the kinds of questions that can be asked about that simple equation:

The question	Replace the z with a number	Comment
If $z = 1$ what is y ?	$1 + 9 = y$	y is now 10.
If $z = 100$ what is y ?	$100 + 9 = y$	y is now 109.
If z is always greater than zero, which describes y ?	Now z is playing the role of all positive numbers.	y will never be smaller than 9 and always positive.

Both z and y are now called “variables.” That makes sense; they do vary! But z happens first and y *depends on* z .

Zero and one (0 and 1) are just numbers. But they are special.

Here are more examples of “when letters stand for numbers”:

Example	Result	Comment 1	Comment 2
$a \times b = y$ when $a=0$.	y is 0.	y will ALWAYS be 0.	$ab = y$ is same question
$a + b = y$ when $a=0$.	y is b .	y will ALWAYS be b .	Think: added nothing
If $7 + x = 7$ then	x is 0.	If $x=0$ then $a + x = a$	and will always be a .
$z \times a = y$ when $a=1$.	$y = z$ EXCEPT	if $z = 0$ then $y = 0$ BUT	If z is NOT zero and $a=1$ then $z \times a = z$.
$\frac{a}{a} = 1$	In fact, if a is NOT equal to zero then	$\frac{\text{any number}}{\text{any number}} = 1$	Repeating: “any number” cannot be zero.

As children, you learned the alphabet. The alphabet was needed to make words. Words were needed to make reading.

The information in this section is alphabet of pre-algebra. Here is the kind of “Show ‘em you know!” test of the information. A test like this one could happen as early as the end of 3rd grade or beginning of 4th grade.

<p>1. What is the value of z in below?</p> $(7 \times 9) \times 3 = (3 \times 7) \times z$ <p>A. 0 B. 1 C. 9 D. z</p>	<p>2. Which of these is NOT true?</p> <p>A. $(a + b) + c = c + (a + b)$ B. $a - b = b - a$ C. $a + b = b + a$ D. $a \times b = b \times a$</p>	<p>3. What is the value of y in</p> $y = 20 \div 5 + 5$ <p>A. 1 B. 2 C. 9 D. 30</p>
<p>4. If $24 \div c = 6$ then what is the value of c?</p> <p>A. 0 B. 3 C. 4 D. 18</p>	<p>5. What is the value of y in the equation</p> $y = 2(7-5) + 10 \div 10$ <p>A. 4 B. 5 C. 6 D. 22</p>	<p>6. Which one of these four is NOT equal to the others?</p> <p>A. $12 \div 3 =$ B. $\frac{12}{3} =$ C. $12 + 3 =$ D. $3 \sqrt{12} =$</p>

<p>7. What should replace the \square in the number sentence below?</p> $100 + 100 + \square = 2 \times 100$ <p>A. 0 B. 1 C. 2 D. 100</p>	<p>8. What is the value of x in $x = 2(11 - 1) + 8 - 16 \div 4$?</p> <p>A. 7 B. 7.5 C. 18 D. 24</p>	<p>9. What should replace the \square in the number sentence below?</p> $(4 \times \square) + 2 = 6$ <p>A. 0 B. 1 C. 3 D. 7</p>
<p>10. If $17 - 8 = 9$, which of the following is a part of this family of numbers?</p> <p>A. $17 + 9 = 26$ B. $20 - 11 = 9$ C. $9 - 8 = 1$ D. $9 + 8 = 17$</p>	<p>11. Which expression is NOT equal to $4 \times (6 + 8)$?</p> <p>A. $(4 \times 6) + (4 \times 8)$ B. $(4 \times 6) + 8$ C. $24 + 32$ D. $4(8 + 6)$</p>	<p>12. The problem $3 + 3 + 3 + 3 + 3 =$ can also be written as</p> <p>A. $15 - 3$ B. $15 \div 3$ C. 3×3 D. 5×3</p>
<p>13. If $ab = c$, which is part of this family of numbers?</p> <p>A. $a \div c = b$ B. $ac = b$ C. $a + b = c$ D. $bc = a$</p>	<p>14. What must replace x in the equation below?</p> $13x + z = z - 9x$ <p>A. 0 B. 1 C. 4 D. Z</p>	<p>15. If $x - y = 0$ then what must y be equal to?</p> <p>A. x B. 0 C. 1 D. 2</p>

This is brand new stuff. This “new stuff” is hard because it is different. How can a z stand for not only one number but many, many numbers? Weird. Testman did not think that was an easy test. Check your answers. How did you do?

Item	Answer	Here is why that answer is correct.
1	C	With addition and multiplication, order does not matter. So $7 \times 9 \times 3$ is the same as $3 \times 7 \times 9$ so the z had to be a 9. Order matter for division and subtraction but not addition and multiplication.
2	B	This is the same type of question and $a - b$ is NOT equal to $b - a$.
3	C	Multiplication and division before adding and subtracting so you should have divided first (got 4) and added that to the 5.
4	C	Family of numbers. If $24 \div c = 6$ then the other two members of the family are $c \times 6 = 24$ and $24 \div 6 = c$. The last family member answered the question.
5	B	Now, since the $(7 - 5)$ is in parenthesis, that happens first. Next comes multiplication and division. That would have given you $y = 2 \times 2 + 10 \div 10$ then $y = 4 + 1 = 5$.
6	C	Family of numbers. Only C does not belong.
7	A	Each side has 200 so the little box can only be zero.
8	D	First: brackets. Second: x and \div so $x = 2 \times 10 + 8 - 4 = 24$.
9	B	$1 \times$ anything is anything so 4×1 is 4 and $4 + 2 = 6$.
10	D	Another family of numbers. For $17 - 8 = 9$, the other two family members are $8 + 9 = 17$ and $17 - 9 = 8$.
11	B	Remember, $4 \times (6 + 8)$ can also be written $4 \times 6 + 4 \times 8$ which makes B wrong.
12	D	5×3 is easier to write, isn't it?
13	A	Another family but no numbers. Bummer! Well, if you have really figured families, then for $ab = c$, the other members are $a \div c = b$ and $c \div b = a$.
14	A	Had to think a little, didn't you? The only value of x that will leave just $z = z$ is $x=0$.
15	A	Still another family of numbers. If $x - y = 0$, then the other members of the family are $x = y + 0$ and $x - 0 = y$.